Layout of symmetric polynomials

$$\begin{tikzcd}
 & \text{Macdonald } (q,t) \\ P_\lambda(x;q,t) \arrow[dl, "q=0"'] \arrow[dr, "t=0"] & \\
\text{Hall-Littlewood } (t) \\ H_\lambda(x;t) \arrow[rr, leftrightarrow, "\text{involution}"] \arrow[dr, "t=0"'] & & q\text{-Whittaker } (q) \\ W_\lambda(x;q) \arrow[dl, "q=0"] \\
 & \text{Schur} \\ s_\lambda(x) &
\end{tikzcd}$$

$$b_\mu(q)\, W_\mu = \sum K_{\lambda\mu}(q)\, s_\lambda$$

Kostka-Foulkes : $s_\lambda = \sum_\mu K_{\lambda\mu}(t)\, H_\mu$

Layout of vertex models

$$\begin{tikzcd}
\text{Higher-spin } A_n \text{ vertex models}^{(t,I)} \\ \text{(symmetric tensor rep. of integer level } I) \arrow[d, "I=1"'] \\
\boxed{\begin{array}{l} A_n \text{ vertex models } (t) \\ sl(2) : \text{six-vertex model} \\ sl(3) : \text{fifteen-vertex model} \\ \vdots \end{array}} \arrow[u, bend right, "\text{fusion [Kirillov-Reshetikhin]}"'] \arrow[d, "\text{degenerations}"] \\
\begin{array}{l} sl(2) : \text{five-vertex model} \\ sl(3) : \text{ten-vertex model (square-triangle)} \\ \vdots \end{array}
\end{tikzcd}$$

Part 1 1. Hecke algebra and polynomial representation

$\Lambda \rightarrow$

$R = \mathbb{Z}[x_1, \dots, x_n]$ — ring of n-variable polynomials

$F = \mathbb{Q}(q,t)$ — field of rational functions in q,t

$R_F = R \otimes_{\mathbb{Z}} F$ — ring of n-variable polynomials with rational coefficients in q,t

$\Lambda_F = \mathbb{Z}[x_1, \dots, x_n]^{S_n} \otimes_{\mathbb{Z}} F$ — ring of symmetric n-variable polynomials, coeff. in q,t.

sym

- The Hecke algebra of type A_{n-1} is generated by $\{T_1, \dots, T_{n-1}\}$ modulo the relations

1) $(T_i - t)(T_i + 1) = 0$, 2) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,

3) $T_i T_j = T_j T_i$, $|i-j| > 1$.

It is a quotient of the braid algebra (by 1)), and after quotienting by a further relation yields the Temperley-Lieb algebra.

- It admits a polynomial representation ρ :

$\rho(T_i) : R_F \longrightarrow R_F$,

given explicitly by

$$\rho(T_i) = t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(1 - \sigma_i), \quad 1 \leq i \leq n-1.$$

Mostly, we abuse notation and write $\rho(T_i) \equiv T_i$.
Operators act to the right!

- The poly. rep. of T_i makes use of the simple transposition

$$\sigma_i \; g(x_1, \dots, x_n) := g(x_1, \dots, x_{i+1}, x_i, \dots, x_n)$$

for any $g \in \Lambda_F$. Introduce another such operator which cyclically permutes variables:

$$\omega \; g(x_1, \dots, x_n) := g(q x_n, x_1, \dots, x_{n-1})$$

The Cherednik-Dunkl operators $\{Y_1, \dots, Y_n\}$ comprise an Abelian subalgebra of the affine A_{n-1} Hecke algebra:

$$Y_i := T_i \cdots T_{n-1} \, \omega \, T_1^{-1} \cdots T_{i-1}^{-1}, \qquad 1 \leq i \leq n$$

$$T_i^{-1} = t^{-1} - t^{-1}\left(\frac{t x_i - x_{i+1}}{x_i - x_{i+1}}\right)(1 - \sigma_i)$$

$$Y_i Y_j = Y_j Y_i \qquad \forall \; i, j.$$

2. (Non) symmetric Macdonald polynomials

- Because the Cherednik-Dunkl operators commute, we can seek to simultaneously diagonalize them:

<u>Def</u> [Cherednik, Macdonald, Opdam] Let $\mu = (\mu_1, \dots, \mu_n)$ be a composition. The non-symmetric Macdonald polynomial (of type A) $E_\mu(x;q,t)$ is the unique solution of

1) $E_\mu = x^\mu + \sum_{\nu \prec \mu} C_{\mu\nu}(q,t)\, x^\nu \quad , \quad x^\nu := \prod_i x_i^{\nu_i}$

2) $Y_i E_\mu = y_i(\mu;q,t) E_\mu \quad , \quad 1 \le i \le n,$

where $y_i(\mu;q,t) = q^{\mu_i} t^{\rho(\mu)_i + n - i + 1}$,

$$\rho(\mu) = -w_\mu \cdot (1, \dots, n), \quad w_\mu \cdot \mu^+ = \mu.$$

- The symmetric Macdonald polynomial $P_\lambda(x;q,t)$ is obtained by symmetrization:

<u>Th^m</u> Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ be a partition and define

$$R^\lambda = \mathrm{Span}_F \{ E_\mu \}_{\mu^+ = \lambda} \subset R$$

There is a unique polynomial P_λ in R^λ such that

1) $P_\lambda = x^\lambda + \sum_{\nu \prec \lambda} d_{\lambda\nu}(q,t)\, x^\nu$

2) $P_\lambda \in \Lambda_F$.

This polynomial is the Macdonald polynomial (of type A) $P_\lambda(x_1, \dots, x_n; q, t)$.

3. Another non symmetric basis [Kusalani–Takeyama]

- Let us define another set of non-symmetric polynomials, $f_\mu(x; q, t)$:

Def Let μ be a composition. The ASEP polynomials (of type A) are the unique family satisfying

1) $f_\delta(x; q, t) = E_\delta(x; q, t)$, $\delta = (\delta_1 \le \dots \le \delta_n)$

2) $f_{\sigma_i \cdot \mu}(x; q, t) = T_i^{-1} f_\mu(x; q, t)$, $\mu_i < \mu_{i+1}$.

i.e. they are built recursively from the anti-dominant composition in each sector.

- A similar symmetrization result holds :

Thm

$$P_\lambda(x; q, t) = \sum_{\mu : \mu^+ = \lambda} f_\mu(x; q, t).$$

- In [Cantini – de Gier – W] the problem of explicitly calculating $f_\mu(x;q,t)$ was addressed; the idea was to seek a formula of the form 6

$$f_\mu(x_1,\dots,x_n;q,t) = \Omega_\mu(q,t)\cdot \mathrm{Tr}_\rho\left[A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\,S\right]$$

where the operators $A_i(x)$, S satisfy the relations

1) $A_i(x)A_i(y) = A_i(y)A_i(x)$,

2) $$t\,A_j(x)A_i(y) - \frac{tx-y}{x-y}\left(A_j(x)A_i(y) - A_j(y)A_i(x)\right) = A_i(x)A_j(y), \quad i<j$$

3) $S\,A_i(qx) = q^i A_i(x)\,S$,

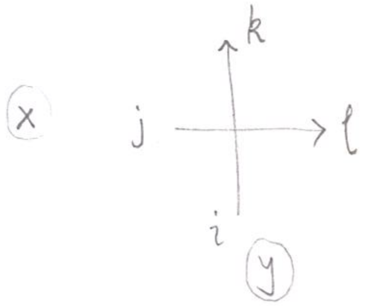
where i,j are discrete ($\mathbb{N}$-valued indices) and x,y are continuous ($\mathbb{C}\otimes F$-valued) indices.

These relations (1) and 2)) are known as the Zamolodchikov – Faddeev algebra.

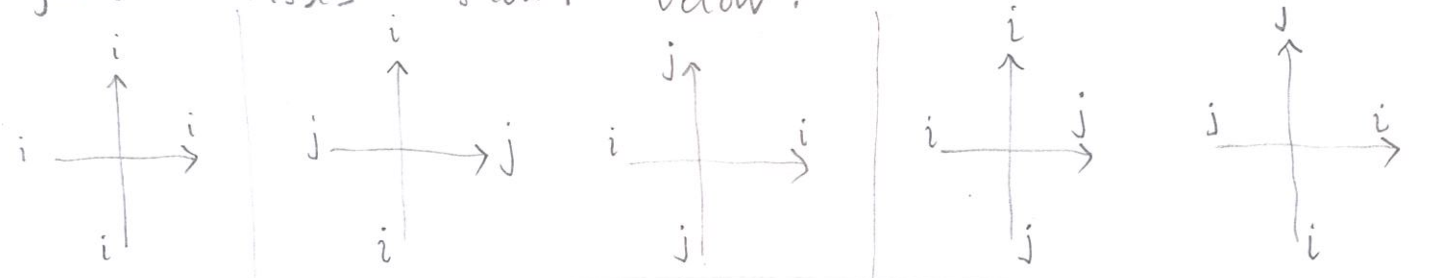
- ρ is some representation of the algebra. The hard work lies in determining ρ.

4. The A_r vertex models

- Let i, j be two numbers in $\{0, 1, \ldots, r\}$. The A_r vertex model is the assignment of a rational function in x, y (called its Boltzmann weight) to pictures of the form

$W_{x/y}(i,j;k,l) :=$  , $\quad i,j,k,l \in \{0,1,\ldots,r\}$.

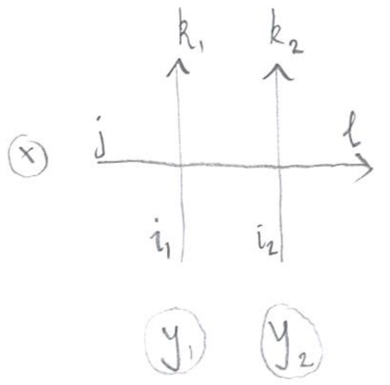
This function is identically zero other than for the five classes shown below:



1	$\dfrac{1-y/x}{1-ty/x}$	$\dfrac{t(1-y/x)}{1-ty/x}$	$\dfrac{1-t}{1-ty/x}$	$\dfrac{(1-t)\,y/x}{1-ty/x}$

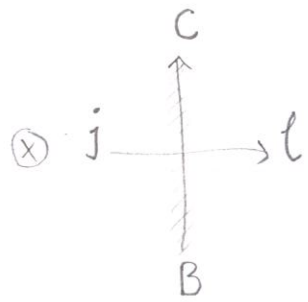
where $0 \leq i < j \leq r$. (Six-vertex model at $r=1$.)

Concatenation of vertices has the following meaning:



$$:= \sum_{m=0}^{r} W_{x/y_1}(i_1, j; k_1, m)\, W_{x/y_2}(i_2, m; k_2, l)$$

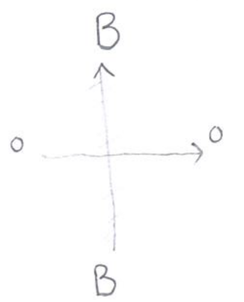
- This vertex model has a "companion" bosonic model. Horizontal edges continue to take values in $\{0,1,\dots,r\}$, but now vertical edges are assigned values $B=(B_1,\dots,B_r)\in\mathbb{N}^r$. The weights:

$L_x(B,j;C,l) :=$ ⓧ 

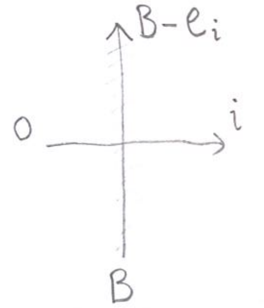
$j,l\in\{0,1,\dots,r\}$

$B,C\in\mathbb{N}^r$

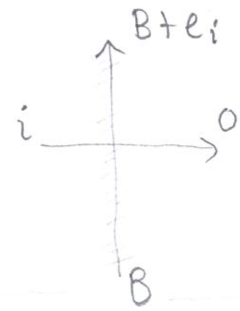
The weights vanish except in the cases indicated below:



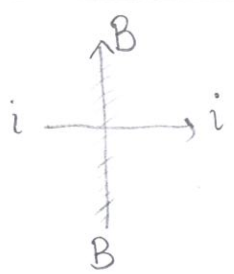
1



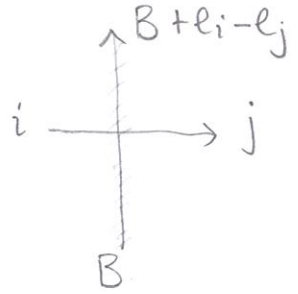
$x(1-t^{B_i})\,t^{B_{(i,r]}}$



1

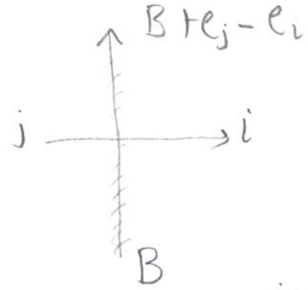


$x\,t^{B_{(i,r]}}$



$i<j$

$x(1-t^{B_j})\,t^{B_{(j,r]}}$



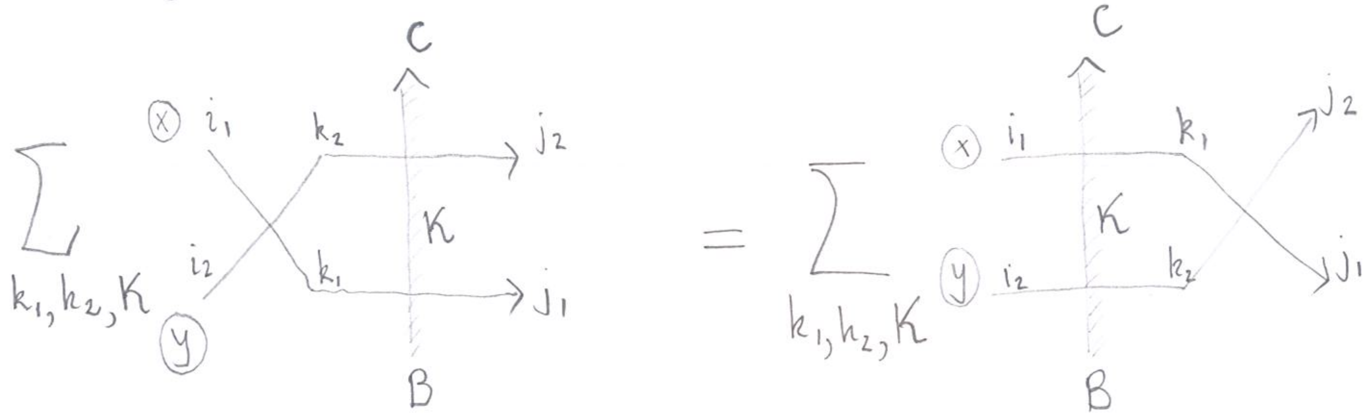
$i>j$

0

- This model can be obtained from the first by fusion, and sending $I\to\infty$.

• The two models are related via the Yang-Baxter equation :

<u>Thm</u> Fix any $i_1, i_2, j_1, j_2 \in \{0, 1, \dots, r\}$ and $B, C \in \mathbb{N}^r$. There holds



or in terms of ω and L,

$$\sum_{k_1, k_2, K} \omega_{x/y}(i_2, i_1; k_2, k_1)\, L_x(B, k_1; K, j_1)\, L_y(K, k_2; C, j_2)$$

$$= \sum_{k_1, k_2, K} L_y(B, i_2; K, k_2)\, L_x(K, i_1; C, k_1)\, \omega_{x/y}(k_2, k_1; j_2, j_1)$$

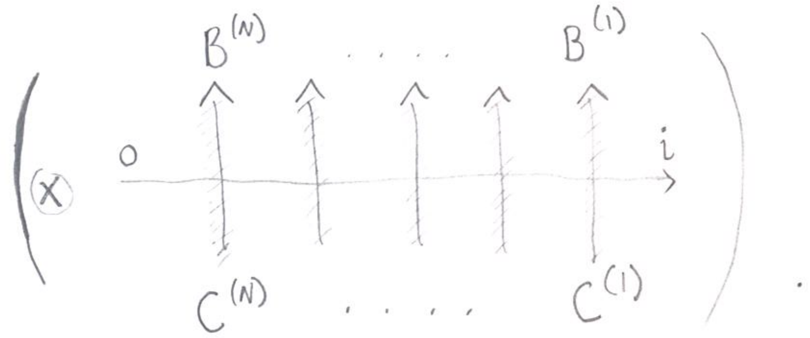
• Now define a vector space $V = \mathrm{Span}_{\mathbb{C}} \{ |B\rangle \}_{B \in \mathbb{N}^r}$ and construct an N-fold tensor product

$\mathbb{V}^{(N)} := V_N \otimes \dots \otimes V_1$, where each V_i is a copy of V.

We define linear operators on $\mathbb{V}^{(N)}$ as follows :

$$A_i(x) : \mathbb{V}^{(N)} \longrightarrow \mathbb{V}^{(N)}$$

$$A_i(x) : |B^{(N)}\rangle \otimes \cdots \otimes |B^{(1)}\rangle \longmapsto \sum_{C^{(1)},\dots,C^{(N)}} |C^{(N)}\rangle \otimes \cdots \otimes |C^{(1)}\rangle$$



<u>Claim</u> The operators $A_i(x)$ are a valid rep. of the ZF algebra 1) and 2)

<u>Proof</u> Using Yang-Baxter equation.

<u>Th^m</u> [CdGW] Let μ be a comp. with largest part r.

$$f_\mu(x;q,t) = \Omega_\mu(q,t) \cdot \mathrm{Tr}_{\mathbb{V}^{(r)}}\left[A_{\mu_1}(x_1) \dots A_{\mu_n}(x_n)\, S(q) \right]$$

where $S(q)$ is a certain diagonal projector factorized over $V^{(r)}, \dots, V^{(1)}$, and

$$\Omega_\mu(q,t) \equiv \Omega_{\mu^+}(q,t) = \prod_{1 \le i < j \le r} \left(1 - q^{j-i} t^{\mu_i^{+\prime} - \mu_j^{+\prime}}\right).$$